2024 18th International Conference on
Control, Automation, Robotics and Vision (ICARCV)
December 12-15, 2024. Dubai

Resilient Consensus for Heterogeneous Networks under Byzantine Attacks

Jie Jiang, Mingde Huang, and Yiming Wu*

2024 18th International Conference on Control, Automation, Robotics and Vision (ICARCV) | 979-8-3315-1849-3/24/$31.00 ©2024 IEEE | DOI: 10.1109/ICARCV63323.2024.10821673

***Abstract*—This paper investigates the consensus problem of multi-agent systems in heterogeneous networks against Byzantine attacks. Heterogeneous networks consist of different subsystems, and existing resilient consensus controllers for heterogeneous networks require specifying specific protocols for each subsystem, with no possibility for interaction between agents in different subsystems. In this study, we present a novel framework, based on hypergraph theory, for representing the complex connections within heterogeneous networks. These networks consist of diverse subsystems, including both discrete and continuous components, and also involve diverse update strategies. Based on the network model, a unified distributed resilient consensus protocol based on Weighted-Mean-Subsequence-Reduced (W-MSR) algorithm is developed and network connectivity conditions are provided in the presence of Byzantine attacks. Finally, the simulation results demonstrate the effectiveness of the proposed algorithm and support the main theoretical findings.**

***Index Terms*—Resilient consensus, heterogeneous networks, hypergraph theory, attack tolerance**

I. Introduction

Coordination in distributed systems has attracted substantial research due to its critical role in ensuring efficient and reliable operations across various fields, such as telecommunication networks [1], smart grids [2], and drone formations [3]. A fundamental topic in this area is the consensus problem, where agents in systems communicate only with their neighbors and reach an agreeable objective. With the advancement of technology, many existing applications are based on consensus in recent years, such as distributed optimization [4], state estimation [5], and robot formation [6].

The traditional consensus problem involves all agents in a homogeneous system adopting a definitive update strategy to achieve consensus [7]. Considerations are typically limited to either continuous systems [8] or discrete systems [9] exclusively. However, real-world network systems are often complex systems formed by the coupling of multiple heterogeneous networks [10], [11], which may be either discrete or continuous. This results in recent research have shifted focus to the consensus problems in heterogeneous systems. The heterogeneity of the system can primarily be reflected in two aspects: (i) the differing nature of subsystems (e.g., discrete or continuous systems); (ii) the variation in update strategies among subsystems.

In [12], the network comprises both continuous and discrete subsystems, and three distinct update strategies are explored to attain consensus. Moreover, [13] expands on this finding and applies it to hybrid second-order systems. In [14], a game model is utilized to construct the system consisting of continuous and discrete subsystems, with update strategies derived from a game-theoretic approach. Switched heterogeneous systems achieving consensus are studied in [15], while [16] focuses on consensus protocols for scaled consensus of switched multi-agent systems.

The aforementioned results do not consider the scenario where the system is under attacks. However, as heterogeneous systems often operate in open and complex environments, there may be different types of attacks like eavesdropping, blocked communication, and false data injection attacks. There are also many countermeasures as detection, strengthening the signal-to-noise ratio, and filter extremum. One of the common attacks is false data injection attack where malicious nodes within the system do not follow the predetermined update strategies, rendering the potential faults and damages to the systems. To address this challenge, some consensus algorithms for countering attacks have been proposed within heterogeneous systems. For instance, [17] investigates the conditions under which consensus can be achieved using the R-censoring strategy in heterogeneous systems composed of discrete and continuous subsystems, in the presence of malicious nodes. [18] studies the consensus of heterogeneous complex networks under attacks. The paper [19] conducts an analysis of a specific class of consensus protocols designed to address Byzantine agents. Additionally, [20] focuses on the study of resilient consensus control in systems that incorporate leaders. Furthermore, both [21] and [22] explore another variant of group consensus in the presence of Byzantine attacks.

Despite the impressive works mentioned, more general scenarios in heterogeneous systems presented with Byzantine agents have been few considered. We consider that each subsystem employs an independent update strategy, and each node may simultaneously be a part of two or more subsystems. This situation is prevalent in the real world, such as when multiple clusters of unmanned aerial vehicles (UAVs) operate cooperatively, leading to variations within different clusters. However, some individual UAVs belong to different clusters at the same time. At this point, the system internally displays a more complex logical topology, which cannot be adequately described using traditional graph theory tools. The hypergraph model, being more generalized than a traditional graph, effectively represents the intricate logical relationships

The authors are with the School of Cyberspace at Hangzhou Dianzi University. Email: ymwu@hdu.edu.cn.

between agents [23]. In this paper, we introduce hypergraph theory, utilizing hyperedges to represent different subsystems, and employ a combination of traditional graph theory and hypergraph theory to describe complex heterogeneous systems. More specifically, the main contributions of this paper can be summarized as follows:

1) We utilize the hypergraph theory to describe the heterogeneity within heterogeneous networks and formulate a unified consensus algorithm framework for such networks.
2) We introduce a characteristic property of coupling in heterogeneous networks and derive sufficient conditions for reaching resilient consensus through the utilization of a specific algorithm known as Weighted-Mean-Subsequence-Reduced (W-MSR), which possesses the coupling properties we have described.

The remainder of the paper is organized as follows: In Section II, we introduce graph theory and hypergraph theory, as well as our system model. In Section III, we present the coupling properties of a specific type of system and analyze the W-MSR algorithm that satisfies these properties, providing the conditions for achieving consensus in heterogeneous systems under the W-MSR algorithm. In Section IV, two numerical simulations are provided to verify our theoretical results and Section V concludes the paper.

Notations: The set of real numbers, positive real numbers, and the set of integers, positive integers are denoted as $\mathbb{R}, \mathbb{R}+$ and $\mathbb{Z}, \mathbb{Z}+$ respectively. $\mathbf{0}$ and $\mathbf{1}$ respectively denote column vectors that are entirely zeros or ones, with adaptive dimensionality. The inner product of two vectors $\boldsymbol{a}, \boldsymbol{b}$ is denoted as $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$. The Hadamard product of two matrices A, B is denoted by $A \odot B$. The notation "$\boldsymbol{a} - \boldsymbol{b} \preceq \mathbf{0}$" indicates that each component of vector $\boldsymbol{a}$ is less than or equal to the corresponding component of vector $\boldsymbol{b}$. "D^+, D_+" represent the upper right and lower right Dini derivatives, respectively. "$\boldsymbol{\omega} = \mathbf{1}(\boldsymbol{a} \preceq \boldsymbol{b})$" denotes the Indicator function, defined as follows:

$$\omega_i = \begin{cases} 0, & \text{if } a_i > b_i, \\ 1, & \text{if } a_i \leq b_i. \end{cases} \tag{1}$$

II. Problem Formulation

A. Preliminaries

Graph theory serves as an effective tool for representing communication networks comprising agents engaged in intercommunication. However, when addressing heterogeneous networks, the integration of hypergraph theory becomes necessary to account for the intricate relationships among sets of agents.

Let $\mathcal{D} = (\mathcal{V},\ \mathcal{E})$ be a directed graph (digraph) on n nodes where $\mathcal{V}$ is the set of agents and $\mathcal{E}$ is the set of communication edges in the network. Adjacency matrix $A = \{a_{ij}\} \in \{0,\ 1\}^{n\times n}$ represents a digraph. If $(i,\ j) \in \mathcal{E}$, we have $a_{ij} = 1$ meaning that agent i can receive message from j. Otherwise we have $a_{ij} = 0$. The set of neighbors of agent i is denoted as $\mathcal{N}_i$.

A hypergraph $\mathcal{H}$ is denoted by $\mathcal{H} = (\mathcal{V};\ E)$ on n nodes where $\mathcal{V}$ is the set of nodes and $E = (e_i)_{i\in I}$ are the subsets of $\mathcal{V}$ called hyperedges (I is a finite set of indexes). Incidence matrix $H = \{h_{ij}\} \in \{0,\ 1\}^{n\times m}$ represents a hypergraph on n nodes and m hyperedges. If $i \in e_j$, we have $h_{ij} = 1$ meaning that agents i in subsystem j. Otherwise we have $h_{ij} = 0$.

Here are some definitions about network robustness in directed graphs.

Definition 1 (r-reachable): [9] Let $r \in \mathbb{Z}_+$ and $\mathcal{D} = (\mathcal{V},\ \mathcal{E})$ be a digraph. A nonempty subset $\mathcal{S}$ of $\mathcal{V}$ is r-reachable, if $\exists i \in \mathcal{S},\ |\mathcal{N}_i \setminus \mathcal{S}| \geq r$.

Definition 2 (r-robust): [9] A digraph $\mathcal{D} = (\mathcal{V},\ \mathcal{E})$ is r-robust, if every pair of nonempty subsets $\mathcal{S}_1,\ \mathcal{S}_2$ of $\mathcal{V}$ satisfies at least one is r-reachable.

Definition 3 ((r, s)-reachable): [9] Let $r, s \in \mathbb{Z}_+$ and $\mathcal{D} = (\mathcal{V},\ \mathcal{E})$ be a digraph. A subset $\mathcal{S}$ of $\mathcal{V}$ is (r, s)-reachable, if $|\chi^r_{\mathcal{S}}| \geq s$ where $\chi^r_{\mathcal{S}}$ is the set of nodes $i \in \mathcal{S}$ satisfying $|\mathcal{N}_i \setminus \mathcal{S}| \geq r$.

Definition 4 ((r, s)-robust): [9] A digraph $\mathcal{D} = (\mathcal{V},\ \mathcal{E})$ is $(r,\ s)$-robust, if for every nonempty pair $\mathcal{S}_1, \mathcal{S}_2$ of $\mathcal{V}$, the following condition is satisfied:

$$(\left|\chi^r_{\mathcal{S}_1}\right| < |\mathcal{S}_1|) \vee (\left|\chi^r_{\mathcal{S}_2}\right| < |\mathcal{S}_2|) \vee (\left|\chi^r_{\mathcal{S}_1}\right| + \left|\chi^r_{\mathcal{S}_1}\right| \geq s). \tag{2}$$

Definition 5: [17] A system $Y = (\mathcal{D}, \mathcal{H})$ is considered $(r_1, \ldots, r_m)$-robust or $[(r_1, s_1), \ldots, (r_m, s_m)]$-robust, where each subsystem Y_i is designated as r_i-robust or (r_i, s_i)-robust.

B. System model

Consider a heterogeneous network system composed of multiple subsystems. Each subsystem in this network is represented as a hyperedge in a hypergraph. Let $(e_i)_{i\in I}$, $I = \{1, 2, ..., m\}$ be the set of subsystems. Furthermore, a heterogeneous system Y is described as $Y = (\mathcal{D}, \mathcal{H})$ where $\mathcal{D}$ indicates the physical layer and $\mathcal{H}$ indicates the logical layer. At the logical layer, the subsystems represented by e_i are coupled to form the hypergraph $\mathcal{H}$. Note that an agent j can belong to more than one subsystem. e.g., $j \in \{e_1,\ e_2\}$.

The updating strategies in different subsystems are independent. The set of updating strategies is denoted by $\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_m\}$. Each agent i in the system maintains a scalar x_i.

The dynamic of discrete-time agent is as follows

$$\begin{cases} x_i(t_{k+1}) = \displaystyle\sum_{j\in(\Omega_i(t_k)\cap e_{p(t_{k+1})})} f^{p(t_{k+1})}_{ij}(x_j(t_k), x_i(t_k)), \\ \quad x_i(t_c) = x_i(t_k), t_c \in (t_k, t_{k+1}) \text{ for all } k \in \mathbb{Z}_+, \end{cases} \tag{3}$$

where the function $f^{p(t)}_{ij}(\cdot)$, $p(t) \in \mathbb{Z}[1, m]$ is related to updating strategy set $\mathcal{U}$ and t_k is the update time step.

The dynamic of continuous-time agent is as follows

$$\dot{x}_i(t) = \sum_{j\in(\Omega_i(t)\cap e_{p(t)})} f^{p(t)}_{ij}(x_j(t), x_i(t)). \tag{4}$$

The switched agent update dynamic between continuous-time and discrete-time is

$$\begin{cases} x_i(t)= \sum\limits_{j\in(\Omega_i(t_{k-1})\cap e_{p(t_k)})} f_{ij}^{p(t_k)}(x_j(t_{k-1}),x_i(t_{k-1})), t\in\{t_k\},\\ \dot{x}_i(t)= \sum\limits_{j\in(\Omega_i(t)\cap e_{p(t)})} f_{ij}^{p(t)}(x_j(t),x_i(t)), \text{ others.} \end{cases} \tag{5}$$

In the paper, we consider two types of agent nodes, namely regular (normal) nodes and Byzantine nodes. The set of regular agents is denoted by $\mathcal{R}$ and the set of Byzantine agents are denoted by $\mathcal{A}$, respectively. Then we give the attack model as follows:

Definition 6 (F-local): [9] A system is a F-local model if $|\mathcal{N}_i \cap \mathcal{A}| \le F$ for all $i \in \mathcal{R}$.

Definition 7 ($(F_1, F_2, ..., F_m)$-bounded): [17] A system is a $(F_1, F_2, ..., F_m)$-bounded model if each subsystem Y_j is F_j-local model.

For analysis convenience, we have the following assumptions:

Assumption 1: Any two discrete subsystems with common agents cannot update simultaneously.

Assumption 2: All subsystems in the heterogeneous system have their own update dynamics. Update processes in all subsystems are independent with common state value in subsystem cross-cover agent.

Assumption 3: The value of $|t_k - t_{k+1}|$ between updates is bounded for all $k \in \mathbb{Z}$.

The definition of safety completed is as follows:

Definition 8 (Safety completed): Let $m(t)$ or $M(t)$ be the minimum or the maximum value of the regular agents at time step t. A system is safety completed if the following holds for any fixed time $t > 0$ and for all $k \in \mathbb{N}$:

(i) the minimum value $m(t), t \in (t_k, t_{k+1}]$ of all agents in $\mathcal{R}$ satisfies $D_+m(t) \ge 0$;

(ii) the maximum value $M(t), t \in (t_k, t_{k+1}]$ of all agents in $\mathcal{R}$ satisfies $D^+M(t) \le 0$.

The goal of the system considered in this study is to achieve resilient consensus through a common local updating strategy.

Definition 9 (Resilient consensus): [9] A system is reaching resilient consensus if the following holds:

(i) $\lim\limits_{t\to\infty} |x_i(t) - x_j(t)| = 0$ for all agents $i, j \in \mathcal{R}$;

(ii) $x_i(t) \in [m(0), M(0)]$ for all agents $i \in \mathcal{R}$ and $t > 0$.

III. Consensus conditions for applying W-MSR algorithm

A. The description of W-MSR algorithm in heterogeneous systems

We first introduce W-MSR algorithm applied in subsystems. Each subsystem represented by a hyperedge independently applies the W-MSR algorithm based on its robustness as described in Algorithm 1. The algorithm only describes the update rules at the moments $\{t_k\}, k \in \mathbb{Z}+$ of update; the state values remain unchanged at all other times. It is essential to emphasize that in our approach, we treat the discrete system as if it were continuous, maintaining constant state values at all times except during updates.

Algorithm 1 W-MSR algorithm in subsystems

Input: e_p, F, $\boldsymbol{x}^{e_p}(0)$
Output: $\boldsymbol{x}^{e_p}(t_K)$
$k = 0$, $t = t_k$
while $k < K$ **do**
  for all $i \in \{e_p \cap \mathcal{R}\}$ **do**
    $\Omega_i(t) = \textbf{sort}\left\{x_j^i(t):\ j \in \{\mathcal{N}_i \cap e_p\}\right\}$
    if $|\{\alpha_i(t) : \alpha_i(t) \in \Omega_i(t), \alpha_i > x_i(t)\}| \ge F$ **then**
      Remove F largest values from $\Omega_i(t)$
    else
      Remove all values in $\alpha_i(t)$ from $\Omega_i(t)$
    if $|\{\beta_i(t) : \beta_i(t) \in \Omega_i(t), \beta_i < x_i(t)\}| \ge F$ **then**
      Remove F smallest values from $\Omega_i(t)$
    else
      Remove all values in $\beta_i(t)$ from $\Omega_i(t)$
    $x_i(t_{k+1}) = x_i(t) + \sum_{j\in\Omega_i(t)} a_{ij}x_j(t)$
    $t = t_{k+1}$, $k = k + 1$

B. The analysis of coupled systems

Heterogeneous systems are composed of multiple different subsystems coupled together. During the coupling process, systems formed by coupling subsystems with specific properties maintain those properties unchanged. The following lemma introduces a special coupling property.

Lemma 1: Suppose Assumptions 1-3 are satisfied. A coupled system $Y = (\mathcal{D}, \mathcal{H} = (\mathcal{V}, (e_1, e_2)))$, composed of two safety-compliant subsystems $Y_1 = (\mathcal{D}_1, \mathcal{H}_1 = (\mathcal{V}_1, e_1))$ and $Y_2 = (\mathcal{D}_2, \mathcal{H}_2 = (\mathcal{V}_2, e_2))$, remains safety compliant.

Proof: We suppose that $\mathcal{D} = (\mathcal{V},\ \mathcal{E})$ is composed of $\mathcal{D}_1 = (\mathcal{V}_1,\ \mathcal{E}_1)$ and $\mathcal{D}_2 = (\mathcal{V}_2,\ \mathcal{E}_2)$. Let $\overline{\gamma}_{Y_i}(t)$ and $\underline{\gamma}_{Y_i}(t)$ represent the maximum value and the minimum value of $\mathcal{R}_i$ in system Y_i respectively at time t. The set of agents $j \in \mathcal{V}_i \cap \mathcal{R}_i$ attained the maximum value and minimum value at time t are denoted by $\overline{V}_{Y_i}(t) = \left\{v_i \in \mathcal{V}_i \cap \mathcal{R}_i : x_i(t) = \overline{\gamma}_{Y_i}(t)\right\}$ and $\underline{V}_{Y_i}(t) = \left\{v_i \in \mathcal{V}_i \cap \mathcal{R}_i : x_i(t) = \underline{\gamma}_{Y_i}(t)\right\}$ respectively.

We first consider the maximum value $M(t)$ at time $t,\ t > 0$. By Definition 8 we have $D^+\overline{\gamma}_{Y_1}(t), D^+\overline{\gamma}_{Y_2}(t) \le 0$. For convenience, we prove the lemma from three cases. *Case 1*: $\overline{V}_Y(t) \cap \mathcal{V}_1 = \varnothing$. That indicates $\overline{V}_{Y_1}(t) \subseteq \mathcal{V}_2$. If $\exists\epsilon > 0$, $\overline{V}_Y(t+\epsilon) \subseteq \mathcal{V}_2$ after updating, then it is easy to show that $D^+M(t) = D^+\overline{\gamma}_{Y_2}(t) \le 0$. If $\exists\epsilon > 0$, $\overline{V}_Y(t+\epsilon) \cap \mathcal{V}_1 \ne \varnothing$ after updating, that means that $M(t+\epsilon) = \overline{\gamma}_{Y_1}(t+\epsilon) \le \overline{\gamma}_{Y_1}(t)$. We have

$$\begin{aligned} D^+M(t) &= \sup \lim_{\epsilon\to 0^+} \frac{\overline{\gamma}_{Y_1}(t+\epsilon) - \overline{\gamma}_{Y_2}(t)}{\epsilon} \\ &\le \sup \lim_{\epsilon\to 0^+} \frac{\overline{\gamma}_{Y_1}(t) - \overline{\gamma}_{Y_2}(t)}{\epsilon} \\ &= \left(\overline{\gamma}_{Y_1}(t) - \overline{\gamma}_{Y_2}(t)\right) \sup \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}. \end{aligned} \tag{6}$$

While $\overline{V}_Y(t) \cap \mathcal{V}_1 = \varnothing$, we have $\overline{\gamma}_{Y_1}(t) < \overline{\gamma}_{Y_2}(t)$ bringing in (6) which yields $D^+M(t) = -\infty$. *Case 2*: $\overline{V}_Y(t) \cap \mathcal{V}_2 = \varnothing$. It is the same as *Case 1*. *Case 3*: $\overline{V}_Y(t) \cap \mathcal{V}_1 \neq \varnothing$ and $\overline{V}_Y(t) \cap \mathcal{V}_2 \neq \varnothing$. That means that $\overline{\gamma}_Y(t) = \overline{\gamma}_{Y_1}(t) = \overline{\gamma}_{Y_2}(t)$. If $\exists \epsilon > 0$, $\overline{V}_Y(t+\epsilon) \cap \mathcal{V}_1 = \varnothing$ after updating, we have $\overline{V}_Y(t+\epsilon) = \overline{V}_{Y_2}(t+\epsilon) \subseteq \mathcal{V}_2$. It is obvious that $\overline{\gamma}_Y(t) = \overline{\gamma}_{Y_2}(t)$ and $\overline{\gamma}_Y(t+\epsilon) = \overline{\gamma}_{Y_2}(t+\epsilon)$. Cause $D^+\overline{\gamma}_{Y_2}(t) \leq 0$, we have $D^+M(t) = D^+\overline{\gamma}_{Y_2}(t) \leq 0$. And it is same as $\exists \epsilon > 0$, $\overline{V}_Y(t+\epsilon) \cap \mathcal{V}_2 = \varnothing$. If $\exists \epsilon > 0$, $\overline{V}_Y(t+\epsilon) \cap \mathcal{V}_1 \neq \varnothing$ and $\overline{V}_Y(t+\epsilon) \cap \mathcal{V}_2 \neq \varnothing$ after updating, then following $D^+\overline{\gamma}_{Y_1}(t), D^+\overline{\gamma}_{Y_2}(t) \leq 0$ it is natural that $D^+M(t) \leq \min\left(D^+\overline{\gamma}_{Y_1}(t), D^+\overline{\gamma}_{Y_2}(t)\right) \leq 0$.

Next, we consider the minimum value $m(t)$ at time t. Recall that

$$\begin{aligned} D_+m(t) &= \inf \lim_{\epsilon \to 0^+} \frac{m(t+\epsilon) - m(t)}{\epsilon} \\ &= \inf \lim_{\epsilon \to 0^+} \frac{\underline{\gamma}_Y(t+\epsilon) - \underline{\gamma}_Y(t)}{\epsilon}. \end{aligned} \tag{7}$$

The rest is similar to the proof of the maximum value $M(t)$, which we can get $D_+m(t) \geq 0$. The proof of Lemma 1 is complete. ∎

Clearly, Lemma 1 states that when a system, composed of several safety completed systems, updates, the values of normal nodes will not exceed the convex set of their initial state values. Then we have the following lemma.

Lemma 2: A system applying W-MSR algorithm is safety completed.

It is obvious that Lemma 2 holds. And it indicates that a system, coupled with subsystems each employing the W-MSR algorithm, is an instance of a safety system. This is one of the reasons we discuss this algorithm. Next, we will analyze the consensus conditions of the proposed algorithm.

C. Consensus Conditions

We define the set of strategies $\mathcal{U} = \left\{\mathcal{U}_1^{F_1}, \mathcal{U}_2^{F_2}, ..., \mathcal{U}_m^{F_m}\right\}$ where $\mathcal{U}_i^F$ represents the W-MSR algorithm with parameter F used by the i-th subsystem. Then we have the following conclusion.

Theorem 1: Let $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ be a digraph with logical partition $\mathcal{H} = (\mathcal{V};\ E = (e_i)_{i \in I})$ for $I = \{1, 2, ..., m\}$. The incidence matrix of $\mathcal{H}$ and the i^{th} column vector are denoted by H and $\boldsymbol{e}_i$ respectively. Let $\boldsymbol{\omega}_i$ represent $\mathbf{1}\left((H\mathbf{1} - \boldsymbol{e_i}) \odot \boldsymbol{e_i} \succeq (r_i+1)\mathbf{1}\right)$. Suppose that the network applies resilient strategy of $\mathcal{U} = \left\{\mathcal{U}_1^{F_1}, \mathcal{U}_2^{F_2}, ..., \mathcal{U}_m^{F_m}\right\}$. Then in the $(F_1, F_2, ..., F_m)$-bounded model, resilient consensus is reached, if $\mathcal{D}$ is $(2F_1+1, 2F_2+1, ..., 2F_m+1)$-robust and $\langle \boldsymbol{\omega}_i, \boldsymbol{\omega}_i \rangle \geq F_i + 1$ for $i \in I$. Furthermore, a necessary condition is $\mathcal{D}$ is $[(F_1+1, F_1+1), (F_2+1, F_2+1), ..., (F_m+1, F_m+1)]$-robust and $\langle \boldsymbol{\omega}_i, \boldsymbol{\omega}_i \rangle \geq F_i + 1$ for $i \in I$.

Proof: For $t > 0$, we define $\zeta(t) = M(t) - m(t) \geq 0$. By Lemmas 1 and 2, it can be shown that D is a safety completed system and $D^+\zeta(t) \leq 0$.

We first prove that $\lim_{t\to\infty} D^+\zeta(t) = 0$. Recall that functions $u, \delta : \mathbb{R} \to \mathbb{R}$ are Heaviside step function and Dirac delta function respectively. The system can be described as

$$\begin{aligned} \zeta(t) &= \zeta(0) + \sum_{i=1}^{\psi(t)} \left[p_i u(t - \sum_{j=1}^{i} t_j) \right] \\ &= \int_{-\infty}^{t} \sum_{i=0}^{\psi(t)} \left[p_i \delta(z - \sum_{j=0}^{i} t_j) \right] dz, \end{aligned} \tag{8}$$

where

$$p_i = \begin{cases} \zeta(0), i = 0 \\ \zeta(t_i) - \zeta(t_{i-1}), i \geq 1. \end{cases}$$

If $\lim_{t\to\infty} D^+\zeta(t) \neq 0$, there exists $\epsilon_0 > 0, \eta_0 > 0$, and $\{\Theta_p\}_{p \in \mathbb{N}} \in \{t_i\}$ where $\lim_{p\to\infty} \Theta_p = +\infty$, $D^+\zeta(\Theta_p) \leq -\epsilon_0$ and $(\Theta_{p+1} - \Theta_p) = \eta_0$. And the following holds:

$$\begin{aligned} &\frac{\zeta(\Theta_{p+1}) - \zeta(\Theta_p)}{\Theta_{p+1} - \Theta_p} \leq -\epsilon_0 \\ &\zeta(\Theta_{p+1}) \leq \zeta(\Theta_p) - \epsilon_0 \left(\Theta_{p+1} - \Theta_p\right) \\ &\qquad \leq \zeta(\Theta_p) - \epsilon_0 \eta_0. \end{aligned} \tag{9}$$

Let $0 < \eta_0 < \epsilon_1$ and $\eta_0 = \frac{\epsilon_1}{2}$, (9) can be expressed as

$$\begin{aligned} \zeta(\Theta_p + \frac{\epsilon_1}{2}) &\leq \zeta(\Theta_p) - \epsilon_0 \frac{\epsilon_1}{2} \\ \zeta(\Theta_p + 2\frac{\epsilon_1}{2}) &\leq \zeta(\Theta_p) - 2\epsilon_0 \frac{\epsilon_1}{2} \\ &\vdots \\ \zeta(\Theta_p + n\frac{\epsilon_1}{2}) &\leq \zeta(\Theta_p) - n\epsilon_0 \frac{\epsilon_1}{2}. \end{aligned} \tag{10}$$

It can be shown that

$$\begin{aligned} \lim_{n\to\infty} \zeta(\Theta_p + n\frac{\epsilon_1}{2}) &\leq \lim_{n\to\infty} \left[\zeta(\Theta_p) - n\epsilon_0 \frac{\epsilon_1}{2}\right] \\ &= -\infty. \end{aligned} \tag{11}$$

This conflicts with $\zeta(t) \geq 0$. Therefore, $\lim_{t\to\infty} D^+\zeta(t) = 0$ must be true.

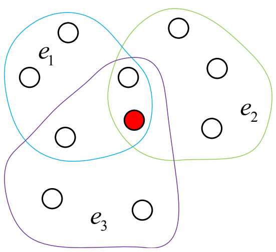
Next we prove that $\overline{\varphi} = \underline{\varphi}$ where $\overline{\varphi} = \lim_{t\to\infty} \overline{\gamma}(t)$ and $\underline{\varphi} = \lim_{t\to\infty} \underline{\gamma}(t)$.

(Necessity) If $\mathcal{D}$ is not $[(F_1+1, F_1+1), (F_2+1, F_2+1), ..., (F_m+1, F_m+1)]$-robust, then there are disjoint, nonempty set $\mathcal{S}_1, \mathcal{S}_2 \subset e_i$ such that condition in Definition 4 is not satisfied. Suppose $x_i(t) = a, x_j(t) = b$ and $a \neq b$ for all $i \in \mathcal{S}_1$ and $j \in \mathcal{S}_2$. Since $\left|\chi_{\mathcal{S}_1}^{F_i+1}\right| + \left|\chi_{\mathcal{S}_2}^{F_i+1}\right| \leq F_i$, suppose all nodes in $\chi_{\mathcal{S}_1}^{F_i+1}$ and $\chi_{\mathcal{S}_1}^{F_i+1}$ are in $\mathcal{A}$ and keep their value constant. The rest of nodes in $\mathcal{R}$ remove the F_i or less values of their neighbors outside $\mathcal{S}_1$ or $\mathcal{S}_2$ so that $\overline{\varphi} = \underline{\varphi}$ cannot reach. If $\mathcal{D}$ is not $\langle \boldsymbol{\omega}_i, \boldsymbol{\omega}_i \rangle \geq F_i + 1$ for $i \in I$, there is a set e_i satisfying $\left|\chi_{e_i}^{F_i+1}\right| < F_i + 1$. Suppose $x_i = a, x_j(t) = b$ and $a \neq b$ for all $i \in e_i$ and $j \in \mathcal{V} \setminus e_i$. Since $\langle \boldsymbol{\omega}_i, \boldsymbol{\omega}_i \rangle < F_i + 1$, suppose all nodes in $\chi_{e_i}^{F_i+1}$ are in $\mathcal{A}$ and keep their value constant. The rest of the nodes in $\mathcal{R}$ remove the F_i or less values of their neighbors outside e_i so that $\overline{\varphi} = \underline{\varphi}$ cannot be reached.

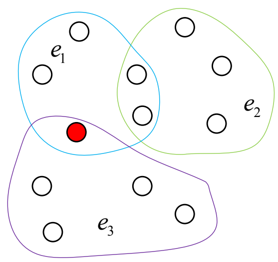
(Sufficiency) Suppose that $\overline{\varphi} \neq \underline{\varphi}$. We have known that $D^+\zeta(t) \le 0$. Since $\mathcal{D}$ is $(2F_1+1, 2F_2+1, ..., 2F_m+1)$-robust and $\langle \boldsymbol{\omega}_i, \boldsymbol{\omega}_i \rangle \ge F_i + 1$ for $i \in I$, each node j^1 or j^2 in any pair of nonempty, disjoint sets $(\mathcal{S}_1 \cap \mathcal{R}), (\mathcal{S}_2 \cap \mathcal{R}) \subset e_i$ is satisfying $\Omega_{j^1}(t) \setminus \mathcal{S}_1 \neq \varnothing$ or $\Omega_{j^2}(t) \setminus \mathcal{S}_2 \neq \varnothing$ in any time $t > 0$. Since $\sum_{j=1}^{N} a_{ij} = 1$ and $a_{ii} = 0$ for all $i \in \{1, 2, ..., N\}$. It can be shown that $\lim_{t\to\infty} D^+\zeta(t) \neq 0$ which conflicts $\lim_{t\to\infty} D^+\zeta(t) = 0$.

The preceding content has demonstrated that each subsystem will not stop updating its state values until the consensus is achieved. Furthermore, $\mathbf{1}\left((H\mathbf{1} - \boldsymbol{e_i}) \odot \boldsymbol{e_i} \succeq (r_i + 1)\mathbf{1}\right)$ ensures that the subsystem with the maximum or minimum state value will definitely receive external information inputs that have not been filtered out. As the system updates iterate, the overall consensus of the system will eventually be achieved. ■

Remark 1: Theorem 1 provides the logical ($\mathcal{H}$) and physical topological ($\mathcal{D}$) conditions required for achieving consensus in heterogeneous systems. The logical topological conditions are described by hypergraphs, while the physical topological conditions are characterized by the robustness of the graph.

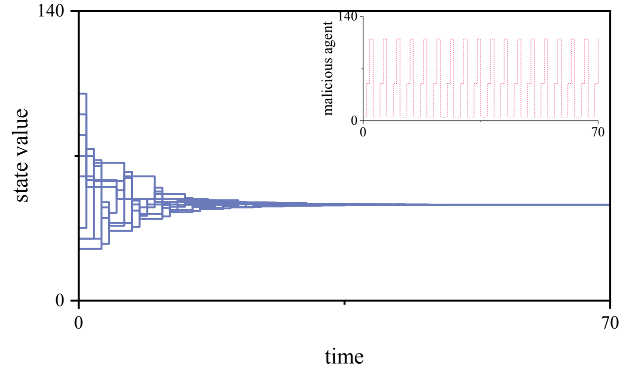


(a) System Y_1.

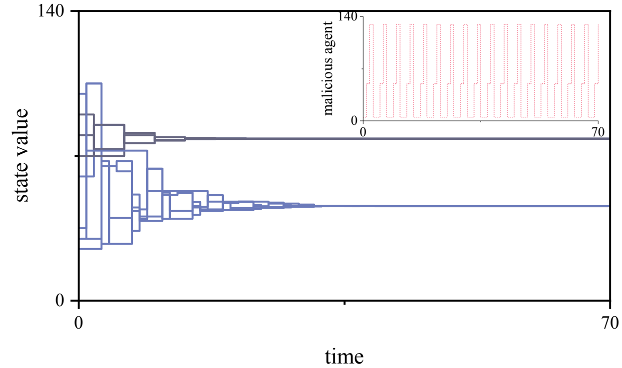


(b) System Y_2.

Fig. 1: The topological representations of system Y_1 and system Y_2 include hyperedges with fully connected vertices. For clarity, the specific connections are not depicted here. The red nodes represent Byzantine agents.



(a) The simulation results of system Y_1.



(b) The simulation results of system Y_2.

Fig. 2: Graphical results of simulation experiments for system Y_1 and system Y_2. The horizontal axis represents time, while the vertical axis represents state values. The main chart shows the state value changes of normal nodes, and the inset chart depicts the state value changes of Byzantine agents.

IV. Simulation Results

In this section, we use two simulation experiments to validate the conclusions of this paper. We consider network $Y_1 = (\mathcal{D}_1, \mathcal{H}_1)$ with $\mathcal{D}_1 = (\mathcal{V}_1, \mathcal{E}_1)$, $\mathcal{H}_1 = (\mathcal{V}_1; E_1)$ and network $Y_2 = (\mathcal{D}_2, \mathcal{H}_2)$ with $\mathcal{D}_2 = (\mathcal{V}_2, \mathcal{E}_2)$, $\mathcal{H}_2 = (\mathcal{V}_2; E_2)$. $\mathcal{D}_1$ and $\mathcal{D}_2$ are both digraphs. The subgraphs of $e_i \subset (E_1 \cup E_2)$ and associated edges are all completed graphs. The robust result of checking all subgraphs mentioned is 3-robust. The initial values of the agents are set to random values within the range of 0 to 120. Each subsystem represented by a hyperedge independently employs W-MSR algorithm for updates.

Simulation 1: We consider the overall system Y_1 as one composed of three subsystems e_1, e_2, e_3 that are coupled together as Fig. 1a. Each subsystem consists of a complete graph with five agents, making them all 3-robust. The attack model of the system satisfies $(1, 1, 1)$-bounded model. Red nodes are designated as Byzantine nodes, which send fixed, distinct values to each different subsystem throughout the update process. The coefficient matrix a for each subsystem is set as follows:

$$a = \begin{bmatrix} 0 & 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 & 0 \end{bmatrix} \quad (12)$$

Simulation 2: The design of simulation 2 is similar, but it differs in that there are changes in the system's topology and the positions of the Byzantine agent have also changed as Fig. 1b.

The simulation results are shown in Fig. 2. More specifically, the results for system Y_1 are shown in Fig. 2a, and the results for system Y_2 are shown in Fig. 2b. As can be observed, system Y_1 satisfies the Theorem 1 and thus ultimately achieves consensus, whereas system Y_2 does not meet the Theorem 1's conditions and therefore cannot guarantee to achieve consensus. Through simulations, we have verified the Theorem 1.

V. Conclusions

In this paper, we have introduced a novel framework using hypergraph theory to model the complex interactions and dependencies inherent in multi-agent consensus networks. By introducing the W-MSR algorithm, we demonstrated how different update strategies can be integrated and made the considered network model resilient against Byzantine attacks. In future work, we will explore more general update strategies, taking into account the presence of time delays. Furthermore, we will analyze the convergence rates of general algorithms within the proposed framework.

References

[1] S. Jere, Y. Song, Y. Yi, and L. Liu, "Distributed learning meets 6g: A communication and computing perspective," *IEEE Wireless Communications*, vol. 30, no. 1, pp. 112–117, 2023.

[2] J. Li, G. Wang, X. Wang, and Y. Du, "Smart charging strategy for electric vehicles based on marginal carbon emission factors and time-of-use price," *Sustainable Cities and Society*, vol. 96, p. 104708, 2023.

[3] J. Xi, C. Wang, L. Wang, M. Qin, K. Xia, and Y. Zheng, "Optimal formation strategy for interconnected quadrotor drones: Theory and experiment," *Aerospace Science and Technology*, vol. 144, p. 108771, 2024.

[4] J. Xu, Y. Tian, Y. Sun, and G. Scutari, "Distributed algorithms for composite optimization: Unified framework and convergence analysis," *IEEE Transactions on Signal Processing*, vol. 69, pp. 3555–3570, 2021.

[5] A. Mitra and S. Sundaram, "Secure distributed state estimation of an lti system over time-varying networks and analog erasure channels," in *2018 Annual American Control Conference (ACC)*, Conference Proceedings, pp. 6578–6583.

[6] H. Park and S. A. Hutchinson, "Fault-tolerant rendezvous of multirobot systems," *IEEE Transactions on Robotics*, vol. 33, no. 3, pp. 565–582, 2017.

[7] H. Ishii, Y. Wang, and S. Feng, "An overview on multi-agent consensus under adversarial attacks," *Annual Reviews in Control*, vol. 53, pp. 252–272, 2022.

[8] X. Zong, T. Li, and J.-F. Zhang, "Consensus conditions of continuous-time multi-agent systems with time-delays and measurement noises," *Automatica*, vol. 99, pp. 412–419, 2019.

[9] H. J. LeBlanc, H. Zhang, X. Koutsoukos, and S. Sundaram, "Resilient asymptotic consensus in robust networks," *IEEE Journal on Selected Areas in Communications*, vol. 31, no. 4, pp. 766–781, 2013.

[10] Y. Susuki, T. J. Koo, H. Ebina, T. Yamazaki, T. Ochi, T. Uemura, and T. Hikihara, "A hybrid system approach to the analysis and design of power grid dynamic performance," *Proceedings of the IEEE*, vol. 100, no. 1, pp. 225–239, 2012.

[11] Y. Shang, "Hybrid consensus for averager–copier–voter networks with non-rational agents," *Chaos Solitons Fractals*, vol. 110, pp. 244–251, 2018.

[12] Y. Zheng, J. Ma, and L. Wang, "Consensus of hybrid multi-agent systems," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 29, no. 4, pp. 1359–1365, 2018.

[13] Y. Zheng, Q. Zhao, J. Ma, and L. Wang, "Second-order consensus of hybrid multi-agent systems," *Systems Control Letters*, vol. 125, pp. 51–58, 2019.

[14] J. Ma, M. Ye, Y. Zheng, and Y. Zhu, "Consensus analysis of hybrid multiagent systems: A game-theoretic approach," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 6, pp. 1840–1853, 2019.

[15] Y. Zheng and L. Wang, "Consensus of switched multiagent systems," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 63, no. 3, pp. 314–318, 2016.

[16] Y. Shang, "Scaled consensus of switched multi-agent systems," *IMA Journal of Mathematical Control and Information*, vol. 36, no. 2, pp. 639–657, 2019.

[17] Y. Shang, "Consensus of hybrid multi-agent systems with malicious nodes," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 67, no. 4, pp. 685–689, 2020.

[18] P. Wang, G. Wen, X. Yu, W. Yu, and Y. Wan, "Synchronization of resilient complex networks under attacks," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 2, pp. 1116–1127, 2021.

[19] D. Angeli and S. Manfredi, "A resilient consensus protocol for networks with heterogeneous confidence and byzantine adversaries," *IEEE Control Systems Letters*, vol. 6, pp. 494–499, 2022.

[20] W. Yue, Y. Yang, and W. Sun, "Resilient consensus control for heterogeneous multiagent systems via multiround attack detection and isolation algorithm," *IEEE Transactions on Industrial Informatics*, vol. 20, no. 3, pp. 4735–4744, 2024.

[21] Y. Shang, "Resilient group consensus in heterogeneously robust networks with hybrid dynamics," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 2, pp. 1456–1469, 2021.

[22] H. Y. Öksüz and M. Akar, "Resilient group consensus in the presence of byzantine agents," *International Journal of Control*, vol. 94, no. 3, pp. 807–822, 2021.

[23] A. Bretto, "Hypergraph theory," *An introduction. Mathematical Engineering. Cham: Springer*, 2013.